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SYMBOLIC ALGEBRA

OR

THE ALGEBRA OF ALGEBRAIC NUMBERS,

Together with Critical Notes on the Methods of Reasoning Employed in Geometry.

BY

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PREFACE.

The object of the following essay is the discussion of negative quantities as found in algebra, or rather the finding a logically developed system, which shall include such quantities as special cases, and thus tend to the great generalization of problems and theorems.

In the older algebras the fundamental theorems were established on the supposition that the various letters, used as symbols, denoted pure *arithmetical* or *absolute* numbers, and the results were *assumed* to be general or true for the so-called negative quantities as well as ordinary numbers. But there are so many objections to a method without a rational basis, even though it gives true results, admitting of a proper interpretation, that of late years attempts have been made to substitute an algebraic series of numbers,

involving both plus and minus quantities, for the ordinary series, so that a definite meaning may be given to a minus number, and the ordinary algebraic processes be given a rational basis for all cases. We shall endeavor in what follows to deduce the ordinary laws of addition, subtraction, multiplication and division, for such a series, calling especial attention to difficulties which naturally occur to any one in the development of the common system, and pointing out how these difficulties are overcome by the "symbolical algebra," as it has been termed.

SYMBOLIC ALGEBRA,

OR

THE ALGEBRA OF ALGEBRAIC NUMBERS.

1. It will conduce to a better appreciation of the subject if we first give very briefly the usual method of deducing the ground rules for arithmetical numbers and point out their limitations.

We shall first then regard the letters used as symbols to denote ordinary numbers.

2. *Addition.*—Let it be required to add $(8a-9b)$ to $(5b-4a)$.

It is tacitly assumed that $(8a-9b)$ is positive and denotes a real number, also that $(5b-4a)$ is positive and denotes an ordinary arithmetical number, whole or fractional. That is, we assume that if the proper numerical values are substituted for a and b, that $8a$ is greater than $9b$, and that $5b$ is numerically greater than $4a$.

Now, if we simply add $5b$ to $(8a-9b)$, we have $(5b+8a-9b)$, but this sum is too great by $4a$, as we had to add $5b$ diminished by $4a$, so that we must subtract $4a$ from this first result to get the correct sum, which is therefore,

$$5b+8a-9b-4a.$$

In order to reduce this expression to its fewest terms, we have to make use of the *law*, that is easily proved for numbers, that the order in which we combine the terms is immaterial. Thus, if from $(5b+8a)$ we subtract $9b$, the result is $(8a-4b)$, since we evidently reach the same value by adding $5b$ to $8a$ and then subtracting $9b$, as in simply subtracting $4b$ from $8a$. From $8a-4b$, we have now to subtract $4a$, giving $4a-4b$ for the correct answer.

In practice we set down the terms and add thus:

$$\begin{array}{r} 8a-9b \\ -4a+5b \\ \hline 4a-4b \end{array}$$

But it must be distinctly understood that $-4a$ by itself means nothing, and

that we have only written for convenience like terms under each other, so that $(-4a+5b)$ must be interpreted $(5b—4a)$, which is agreeable to the law mentioned.

From a consideration of such examples we deduce the law in addition: Combine like terms by adding those of like sign and prefixing the common sign, and when of unlike signs take the difference of the sum of the positive and the sum of the negative terms and prefix the sign of the greater.

3. *Subtraction.*—Suppose we have to subtract

$$5b—4a \text{ from } 8a—9b.$$

If we take $5b$ from the minuend, the indicated result is

$$8a—9b—5b;$$

but we have taken away too much by $4a$, for we had only to subtract $5b$ less $4a$, so that we must increase this result by $4a$, giving for the correct answer

$$8a—9b—5b+4a=12a—14b.$$

Combining the terms as mentioned above by adding $8a$ and $4a$ and subtracting

$14b$, which is the same thing as first subtracting $9b$ and then $5b$. This result is briefly represented thus:

$$\begin{array}{rl} \text{From} & 8a - 9b \\ \text{Subtract} & -4a + 5b \\ \hline \text{Remainder} = & 12a - 14b \end{array}$$

which is thus equivalent to the rule; change the signs of the subtrahend and proceed as in addition.

We again note that the minuend and subtrahend are tacitly assumed to be real positive numbers, whole or fractional, and further that the subtrahend is numerically in value less than the minuend. There is, besides, no sense in subtracting $-4a$ from $+8a$ by itself, for $-4a$ has no existence by itself. Neither can we subtract $5b$ from $-9b$ since the last term is an absurdity by itself in arithmetical algebra. Neither can any meaning be attached to adding the same terms ($-4a$ to $8a$ or $5b$ to $-9b$), in article 2, though we get so accustomed to using the well-known rules in apparently adding or subtracting such single terms that

we lose sight of our tacit assumptions, that it is only when combined into two expressions, both numerically positive and in the case of subtraction, the minuend greater than the subtrahend, that we can apply such rules at all.

4. *Multiplication.*

$$(a+b)c=ac+bc$$

$$(a-b)c=ac-bc$$

$$(a+b)(c+d)=(ac+bc)+(ad+bd)$$

$$(a-b)(c+d)=(ac-bc)+(ad-bd)$$

These four results are obvious enough since $(a+b)c$ means the number $(a+b)$, repeated c times, and is equal to a (one part), repeated c times, added to b (the other part), repeated c times. Similarly we go through. In the next case we have to multiply $(a+b)$ by $(c-d)$ or repeat $(a+b)$ c times and diminish the product by $(a+b)$ repeated d times

$$\therefore (a+b)(c-d)=ac+bc-(ad+bd)$$
$$=ac+bc-ad-bd$$

Similarly,

$$(a-b)(c-d)=ac-bc-(ad-bd)$$
$$=ac-bc-ad+bd$$

Now, in place of going through the successive steps given here, we shall reach the same result by multiplying each term of the multiplicand by each term of the multiplier, noting a rule that *like* (prefixed) *signs give plus*, *unlike minus*, in the product. It is again tacitly assumed that both multiplicand and multiplier are positive (arithmetical) numbers.

There is no sense, *per se*, in

$$-b \times c = -bc$$

or in

$$-b \times -d = +bd$$

It is only in connection with the other terms that such results have any meaning so far as expressions containing only arithmetical numbers are concerned.

5. *Division.*—In division the aim is to find a factor called *quotient* that multiplied by the divisor will give the dividend, so that the rules of signs and limitations hold as in multiplication.

6. We see, therefore, that with such an *algebra of ordinary numbers*, connected by addition and subtraction signs, that we are very much circumscribed, for

we have seen that the expressions we treat must be essentially positive, whereas, in numerous investigations, it is impossible to say whether all the expressions we combine by addition, subtraction, multiplication or division, are positive or negative, so that we are, perforce, obliged to apply our rules to negative quantities without having defined them or proved any rules with respect to them.

In certain problems, too, the unknown quantity is found with a negative sign before it, and we have to go back and vary the statement of the problem to make the result intelligible, when that is possible.

The great value that lies in the conception of the true, real, negative quantities, consists in the wonderful generality given to general formulæ, which may be deduced, say, on the supposition that all the letters denote plus quantities, when a simple substitution in the final result of the actual values (in any case) of the letters (for that case) will give a correct formula.

In the applications of algebra to geom-

etry we shall find still stronger reasons for an early consideration of + and — quantities.

We thus see that it is not only necessary in certain cases but highly important that the greatest generality should be given to algebraic processes, and we shall now proceed to the consideration of the algebraic series of numbers, which, together with an extension of meaning given to the signs plus and minus, will enable us to give the generality desired.

7. The algebraic system of numbers is supposed to increase from zero in the opposite directions as in the scale below.

Those numbers to which are prefixed the + sign are called positive numbers, those with the negative sign minus numbers; so that $(-a)$ means a quantity or number of *real* units, but measured in a direction opposite to the positive one. There is no meaning in the statement that $(-a)$ is "less than zero" without a new definition of "less than," so let the reader bear in mind that in dealing with $(-a)$ he is dealing with a real arithmeti-

−4
−3
−2
−1
0
+1
+2
+3
+4

cal number, though taken in the opposite sense to $(+a)$.

For the present it is well to indicate an algebraic number by enclosing it in a parenthesis.

The + and — quantities above may be conceived as measuring distances right and left from 0, so that the line drawn will be considered as divided into equal parts and may be called the scale of numbers. In this case direction to the right is called +, that to the left —, but we can just as well assume the positive direction to the left when the negative way of counting will be to the right.

As illustrations of quantities having opposite directions, we may mention motion forward and backward, or degrees above and below zero on a thermometric scale. Thus, if a moving point, starting at 0 in the scale, moves 5 units of distance to the right and 9 to the left, we reach (-4) on the scale which thus indicates that the final position of the moving point is 4 units to the left of the starting point or zero of the scale.

Again, if it moves 5 units forward, then 9 units backwards, then 6 units forward, its final position will be 2 units from 0 in a forward direction, or to the place denoted by (+2). If direction forwards is called +, then direction backwards will be — as we see illustrated in this case.

8. *Addition of Algebraic numbers.*—*Rule.* To add two algebraic numbers, *we start at the point in the series indicated by the first number and count in the direction given by the sign of the second number the number of absolute units in that number.*

This operation is indicated by putting the + sign between the algebraic numbers; thus to add (+3) to (—5) we start at (+3) in the scale and count in the negative direction 5 units, thus bringing us to (—2). The symbolical representation is as follows:

$$(+3)+(-5)=(-2).$$

Similarly, in "adding" any number of algebraic numbers, we add the first two

as just explained, then the third to the previous result by the same rule, and so on.

Thus,

$$(+3)+(-5)+(+3)=(+1).$$

Here we start at $(+3)$, go 5 units to left to (-2), then 3 units to right to $(+1)$, the result of the *algebraic addition.*

We see that the rule for adding two algebraic numbers is consistent with the definition, *that* $(-a)$ *means absolute units measured in the opposite direction on the scale of numbers to the plus direction, so that when combined by the sign of addition with* $(+a)$, *the result is zero.*

9. By the consideration of examples like the following:

$$(+3)+(+2)=(+5);$$

$$(-3)+(+2)=(-1);$$

$$(+3)+(-2)=(-1);$$

$$(-3)+(-2)=(-5;$$

We easily deduce the rules:

I. To add two numbers with *like* signs,

find the arithmetical sum of their absolute values and prefix the common sign to the sum.

II. To add two numbers with unlike signs, find the arithmetical difference of their absolute values and prefix the sign of the greater.

Where there are many numbers to add of different signs, we place the + numbers in one column, and the − numbers in another column, add separately and combine the sums. It is evident that the result is the same in whatever order we add algebraic numbers, since the total number of units we go to the right, by the successive steps, equals the sum of the absolute values of the positive numbers. Similarly for the minus numbers; so that the final position is given by the relation between the two sums.

10. Let a and b denote the absolute values of any two numbers, then the four cases given in article 9 can be generalized and written in the following forms.

$(+a)+(+b)=a+b$;

$(-a)+(+b)=-a+b.$

$(+a)+(-b)=a-b$;

$(-a)+-b)=-a-b.$

The interpretation to be given to the right members must follow from some convention, and this convention is, that when a quantity, as a, *has no sign before it, the plus sign must be understood*; so that a is the same thing as the algebraic number $(+a)$; likewise, that *if a quantity* a *has* a + *or* − *sign before it, that sign will indicate whether it is* a + *or* − *algebraic number*, to be combined by addition with any other numbers in the same expression.

Thus, $a-b$ is the same as $(+a)+(-b)$, and $-a-b$ the same as $(-a)+(-b)$.

This convention enables us to dispense with parentheses and a multiplicity of signs when desired; though for the sake of clearness we shall retain them in establishing fundamental rules, whilst indicating, at the same time, abreviated forms. However, if an expression is

given in the abbreviated forms, shown by the right members of the above equations, we can always express it in the manner shown by the left members, when desirable.

II. We will now show how the use of algebraic numbers enables us to generalize by expressing all the cases of a certain problem by one equation.

We must always premise by stating what class of quantities are to be regarded as +, and which (those opposed to the + one) minus. Thus, in the case of the moving body in art. 7, let us count distances forward +, those backward, minus.

Then, if a person walks say 5 feet forwards from a given origin, then 9 feet backwards along the same straight line produced when necessary, we can say he moves successively $+5$ and -9 feet, thus stopping 4 feet back of the origin, or at -4. Thus, if we call X his final distance in feet from the origin, + if forwards, − if backwards, we can write,

$$X=(+5)+(-9)=(-4),$$

or, more simply,

$$X=5-9=-4.$$

Again, if the person moves 5 feet forwards, then 9 backwards, then 6 forwards, we have his ultimate position,

$$X=5-9+6=+2;$$

or two feet in front of the origin.

Now, no matter how many movements forwards and backwards are made, we see that we can state generally that the final position of the moving body is the algebraic sum of his successive movements, so that,

$$X=(a)+(b)+(c)+\ .\ .\ .$$

where a, b, c, . . . represent the successive movements which are essentially negative when backwards, otherwise positive, as written.

Again, as to dates, let dates reckoned after Christ be designated as +, those anterior —, and let any interval of time be regarded as + if reckoned forwards, — if reckoned backwards; then having given a date D and the interval I to another date D′, we can find the letter by the formula,

$$D+I=D'.$$

What is the date 20 years after 240 B.C.?

$$(-240)+(20)=-220,$$

or 220 B.C.

What is the date 20 years before 240 B.C.?

$$(-240)+(-20)=-260,$$

or 260 B.C.

What is the date 40 years after 20 B.C.?

$(-20)+(+40)=(+20)$, or 20 A.D.

Julius Cæsar died A.D. 14, aged 77 years; when was he born?

$(+14)+(-77)=-63$, or 63 B.C.

Similarly for any series of successive intervals we shall always have,

$$D+\text{sum of intervals}=D'.$$

Thus one formula comprehends every case. Other illustrations of such generalizations can be made in the case of quantities that stand in opposition, as *assets* and *debts*, *gain* and *loss*, &c. The algebraic sum of assets taken as plus, and sum of debts as minus, gives total assets if the result is +, or total indebtedness if −, &c.

Examples of a far more complicated character will be given further on, as we deduce rules for subtraction, multiplication, &c., of algebraic numbers.

Subtraction.

12. We have previously only used the + sign before an *algebraic* number, having clearly defined the process of addition intended by it. We shall similarly give a definition of a *minus sign* before an *algebraic number*, consistent with the use already made of it to indicate a negative number, and will remark that everything that follows will be strictly deduced from the consequences of these definitions.

Definition.—A minus sign before an algebraic number indicates that the number is to be counted in an opposite direction to that indicated by its sign, or it indicates a reversal of direction of the line representing that number.

Thus,

$$-(+5)=(-5),$$

$$-(-5)=(+5).$$

This process involves no absurdity for

single numbers; for the results (-5) and $(+5)$ in the two cases are perfectly intelligible; meaning algebraic numbers, the one plus, the other minus, that will cancel each other if added together. From the definition, a $-$ sign before an arithmetical number causes it to be counted in the opposite direction to the usual or $+$ direction, as the number may be regarded as a $+$ number.

13. When this minus sign is placed between two numbers, as $(+8)-(+5)$, it indicates what is called *algebraic subtraction.* The quantity before which it is placed is called the subtrahend, the other one the minuend.

From the definition, it follows that to subtract one quantity from another, *we change the sign of the subtrahend and add.*

Thus,

$$(+3)-(+1)=3-1=2,$$
$$(+3)-(-2)=3+2=5,$$
$$(-3)-(+1)=(-3)+(-1)=(-4),$$
$$(-3)-(-1)=(-3)+(+1)=(-2).$$

We have left the parentheses around certain minus quantities in the right numbers, though it is not necessary if we attend to the meaning given in art. 10.

It is seen from a consideration of these cases that the *absolute number* given as the result of the algebraic subtraction of two algebraic numbers, *gives their distance apart on the scale of numbers, and that the prefixed sign is the direction from the subtrahend to the minuend.* Hence the result is called the *algebraic difference.*

As in art. 10, we can write briefly,

$$(+a)-(+b)=a-b\,; \qquad (-a)-(+b)=-a-b,$$

$$(+a)-(-b)=a+b\,; \qquad (-a)-(-b)=-a+b.$$

In the right numbers of these equations the a's and b's are to be taken with the sign prefixed to each, or understood according to the convention given in art. 10, and added by the rules given in art. 9.

14. We see from the foregoing that we employ the same rules exactly as in ad-

dition and subtraction of polynomials in the arithmetical algebra, and that when + single numbers are taken, the algebraic sum or difference of any two of them is exactly as in the arithmetical algebra. As before explained, arithmetical algebra cannot handle single minus numbers or expressions that are essentially minus; but now there are no limitations as to signs, for the operations on negative quantities are just as intelligible and as clearly defined as on positive quantities.

15. On comparing the formulæ just given in art. 13 with those in art. 10, we notice the relation:

$$(+a)+(+b)=(+a)-(-b),$$

$$(+a)+(-b)=(+a)-(+b),$$

or

$$a+b=a-(-b),$$

$$a+(-b)=a-(+b)=a-b.$$

So that algebraic additions can always be written as algebraic subtraction by changing the sign of the term, which is to appear as the subtrahend. We can thus go from one form to the other at

pleasure. In fact, from the comprehensive definition given to a minus sign before an algebraic number (article 12), we can write,

$$a+b=a-(-b)=a-[-(+b)]$$
$$=a-[-(-(-b))], \text{ etc.}$$

Similarly,

$$a-b=a-[-(-b)]$$
$$=a-[-\{-(-(-b))\}].$$

There is thus choice in the form in which a given value can be expressed.

16. The principles deduced for algebraic numbers apply equally to algebraic expressions, composed of several numbers combined in any way, since, if the numerical values are assigned to the letters and the indicated operations performed, the expression reduces to a single number when the rules apply.

Thus to subtract $5b-4a$ from $8a-9b$, we have,

$$8a-9b-(5b-4a)=$$
$$8a-9b-5b+4a=$$
$$12a-14b.$$

Compare the result in art. 3, where it

was premised that $8a$ was greater than $9b$, and that $5b$ was greater than $4a$, also that the subtrahend $(5b-4a)$ was less than the minuend $(8a-9b)$. By the use of algebraic numbers there are no limitations of this kind whatever. As in this example, we shall generally, in what follows, leave off the parentheses to indicate the algebraic numbers. From a consideration of the results in art. 15, the expressions can be transformed so as to contain them when desired.

Multiplication.

17. In multiplying any number by a, we understand that the number is to be repeated a times, so that we only have to define what is meant by multiplying by $+a$ or $-a$; in other words the significance of the $+$ or $-$ signs before a multiplier.

Now, *since* a $+$ sign before an algebraic number does not reverse the direction of counting (art. 10), and a $-$sign does (art. 12), we shall define,

1. Multiplication of an algebraic num-

ber by $(+a)$ means that the number is to be repeated a times, the direction of counting being unchanged.

2. Multiplication of an algebraic number by $(-a)$ signifying that the number is to be repeated a times, and the direction of counting reversed.

Thus,

$$(+a)\times(+b)=+ab,$$
$$(-a)\times(-b)=+ab,$$
$$(+a)\times(-b)=-ab,$$
$$(-a)\times(+b)=-ab.$$

From the first two results we deduce the rule, "*like signs give plus*," and from the last two the rule, that "*unlike signs give minus.*"

18. Since *division* may be defined as finding the factor called quotient, when the other factor (the divisor), and the product of the two factors, called the dividend, is given, it is seen that the same rules for signs hold for division.

Thus,

$$\frac{(+ab)}{+b}=a;\frac{(+ab)}{-b}=-a;$$

$$\frac{(-ab)}{-b}+a;\ \frac{(-ab)}{+b}=-a.$$

19. Since a fraction may be considered an indicated quotient, the same laws as to signs hold for fractions.

20. Having defined multiplication of single numbers by plus and minus quantities, we must now ascertain if these rules can be utilized for the multiplication of polynomials by polynomials.

Now,

$$(a-b)n=(a-b)+(a-b)\ \ldots\ n \text{ times},$$
$$=a+a+\ \ldots\ n \text{ times}, -b-b-\ \ldots\ n \text{ times},$$
$$=na-nb.$$

This is called the distributive law. Similarly, to multiply $(a-b)$ by $-n$, we get the same result by *definition*, except that the direction of counting of the result is changed, so that

$$(a-b)\times -n=-na+nb.$$

Hence the rule: multiply each term of the polynomial by the monomial, according to the laws laid down for single numbers, and add the results. Next suppose

the multiplier, $(+n)$ or $(-n)$, is itself expressed as a polynomial. Thus, if 10 is to be multiplied by $8-3$ or 5, it is evident that we get the same result by repeating the 10 8 times in the positive direction and 3 times in the negative direction, and adding the results as by multiplying the 10 by 5.

This operation is recorded thus:

$$\begin{array}{r} 10 \\ +8-3 \\ \hline +80-30=50 \end{array}$$

That is we multiply the multiplicand by each term of the multiplier according to the laws for single numbers, and add the partial products.

Similarly we multiply 10 by $-8+3$ or -5,

$$\begin{array}{r} 10 \\ -8+3 \\ \hline -80+30=-50 \end{array}$$

Here the direction of the 10 is reversed on multiplying by -8, giving -80; but not reversed on multiplying by the $+3$,

which gives $+30$, which added to -80 gives -50; exactly what we should obtain by repeating the 10, 5 times in the negative direction. There is no difference in principle in multiplying a negative multiplicand by a polynomial.

In fact we are not limited, as in the case of arithmetical numbers, but both multiplicand and multiplier can be essentially negative, and the results are as perfectly intelligible as in the case of absolute numbers. The laws just deduced, by combination enable us to apply the usual rules to the multiplication of any polynomial by another polynomial. Thus to multiply $(-a-b)$ by $(c-d)$, we obtain,

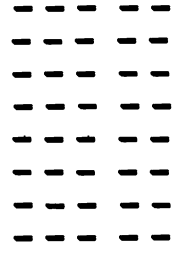
$$\begin{array}{r} -a-b \\ c-d \\ \hline -ac-bc+ad+bd. \end{array}$$

That is, we multiply the $(-a-b)$ first by c, by the distributive law, without reversing, and then by d, reverse the last result, and combine with the other.

21. The rules for the division of poly-

nomials by polynomials, are deduced from these in the usual manner.

22. The rule for multiplying by a polynomial may be illustrated graphically as follows: Let it be required to multiply 8 by $(3+2)$, or generally m by $(a+b)$.



There are 8 units in each vertical row, and 8×3 and 8×2 units in the two columns, making $8\times5=8\,(3+2)$ units in all.

$$\therefore\ 8(3+2)=8\times3+8\times2.$$

Similarly,

$$8(5-2)=8\times5-8\times2\,;$$

for 8×5 gives the whole number of units and $8+2$ those of the right column, so that their difference equals the number in the left column $=8\times3=8(5-2)$.

23. We shall note a special case of multiplication, $(-a)^a$ as the form often occurs. Now by the rule, article 17, we have

$$(-a)^2=(-a)\times(-a)=+a^2$$

for $(-a)^a$, means that $(-a)$ is to be repeated a times and the direction of counting reversed.

If this result is again multiplied by $-a$, by the same rule, we obtain,

$$(-a)^3=+a^2\times-a=-a^3$$

Similarly

$$(-a)^4=(-a^3)\times-a=+a^4;$$

so that a negative quantity raised to an even power gives a + result; to an odd power a — result.

24. We shall group together some of the fundamental formulae so far deduced, and note an important fact resulting from the comprehensiveness of the definitions relating to the influence of signs.

The following formulae have been drawn from articles 10, 13, 17 and 18; and it is to be observed that the left members indicate the addition, subtrac-

tion, multiplication and division of algebraic numbers, and the right members are intended to show the indicated operation *performed*, as far as this can be done where numerical values are not assigned to a and b.

We have put parentheses around the algebraic numbers, in the left members, as before, for without them, in the case of $a-b$, we might be in doubt whether we are to include it under the rule of addition and write it, $(+a)+(-b)$ or to consider it a case of subtraction and write it $(+a)-(+b)$. The parentheses around the plus quantities as well as the sign + enclosed, can be omitted, if desired, since a quantity is assumed as +, when it has no sign before it, so that $a+b$ would generally be understood as $(+a)+(+b)$; but as it fixes the attention more precisely upon the algebraic operations involved, the parentheses were retained in the left members throughout.

The right members are, of course, algebraic members, only under a different form from the left numbers (see ar-

ticle 10), the results of the combinations, according to the rules, in both members, being precisely identical.

The formulae are as follows:

$$1.\quad \begin{cases} (+a)+(+b)=a+b; \\ (-a)+(+b)=-a+b; \\ (+a)+(-b)=a-b; \\ (-a)+(-b)=-a-b; \end{cases}$$

$$2.\quad \begin{cases} (+a)-(+b)=a-b; \\ (-a)-(+b)=-a-b; \\ (+a)-(-b)=a+b; \\ (-a)-(-b)=-a+b; \end{cases}$$

$$3.\quad \begin{cases} (+a)\times+b=+ab; \\ (+a)\times-b=-ab; \\ (-a)\times-b=+ab; \\ (-a)\times+b=-ab; \end{cases}$$

$$4.\quad \begin{cases} \dfrac{(+ab)}{+b}=a; & \dfrac{(+ab)}{-b}=-a; \\ \dfrac{(-ab)}{-b}=a; & \dfrac{(-ab)}{+b}=-a. \end{cases}$$

Now note, that if in the left members we substitute $(-a)$ everywhere for a and perform the indicated operations, as far as may be, that the result is precisely the same as if we substitute $(-a)$ for a

in the corresponding right members, and reduce according to the laws established.

Thus put $(-a)$ for a in both members of the first formulae of each series, we obtain

$$(-a)+(+b)=-a+b$$
$$(-a)-(+b)=-a-b$$
$$(-a)\times(+b)=+(-a)b=-ab$$
$$\frac{(-a)b}{+b}=(-a)=-a.$$

Thus having deduced a formula, as $a\times b=ab$, we can ascertain the result for $(-a)\times b$ by substituting $(-a)$ for a in the right member, or performed multiplication, giving $(-a)b=-ab$.

25. Again to multiply $(a+b)$ by a, we indicate the process as follows:

$$\begin{array}{r} a+b \\ a \\ \hline a^2+ab \end{array}$$

This may be called a general formula, true whether b is $+$ or $-$; for if b is minus, so that we have to multiply $a+(-b)$ by a, we have only to change b to

$(-b)$ in the result and reduce by the laws of signs, giving the correct answer, a^2-ab. This follows from the principle just noted.

In the formula,

$$n(a+b+c)=an+bn+cn,$$

if we put $a=-x$, $b=-y$, in both numbers, the results must be identical by the principle or law that was found to exist.

$$\therefore\ n\,(-x-y+c)=-xn-yn+cn.$$

In the multiplication or division of polynomials by polynomials, the same general principle as to substitution exists, for the results are derived from the multiplication, division, addition and subtraction of monomials according to the rules affecting single numbers.

Thus by actual multiplication, we deduce the formula,

$$(a+b)^2=a^2+2ab+b^2$$

By substituting $(-b)$ for b in this formula, both sides, and reducing, we find the correct formula,

$$(a-b)^2=a^2-2ab+b^2.$$

We can put the actual multiplications side by side to make this plainer:

$$\begin{array}{ll} a+b & a+(-b) \\ a+b & a+(-b) \\ \hline a^2+ab & a^2+(-ab) \\ \quad +ab+b^2 & \quad +(-ab)+(-b)^2 \\ \hline a^2+2ab+b^2 & a^2-2ab+b^2 \end{array}$$

Thus if in the partial product $(a+b)\times a=a^2+ab$ we substitute $(-b)$ for b, we get the first partial product on the right. Similarly for $(a+b)\times b$. Now since, by article 15, we can write $+(-ab)=-ab$, and by article 23, $(-b)^2=+b^2$, the result on the right is $a^2-_2ab+b^2$, just what we find by putting $(-b)$ for b in the final result on the left.

Now it is readily seen that all this follows from the general principle enunciated above, so that it is not necessary to analyze every case in this way, but we may consider that the principle of substituting $(-a)$ for a on both sides of any formula giving identical results as established.

Again having deduced the general formula,

$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$,

if in any particular example b and c say are minus, we have at once without the necessity of multiplication, by substitution only,

$(a-b-c)^2=a^2+b^2+c^2-2ab-2ac+2bc$.

26. The principle in question is likewise evident if the minus quantity is always kept in a parenthesis, thus simply taking the place of the corresponding + quantity throughout. Thus to cube $(a-b)$, we may multiply as follows:

$$\begin{array}{l} a+(-b) \\ a+(-b) \\ \hline a^2+a(-b) \\ \quad +a(-b)+(-b)^2 \\ \hline a^2+2a(-b)+(-b)^2 \\ a+(-b) \\ \hline a^3+2a^2(-b)+a(-b)^2 \\ \quad +a^2(-b)+2a(-b)^2+(-b)^3 \\ \hline a^3+3a^2(-b)+3a(-b)^2+(-b)^3 \\ =a^3-3a^2b+3ab^2-b^3. \end{array}$$

If we had formed the product $(a+b)^5$ along side of this, we should have noted the identity throughout, except that b is everywhere simply replaced by $(-b)$, so that the final result can be obtained from that pertaining to $(a+b)^3$ by simply changing b to $(-b)$ and reducing according to the laws of signs.

Similarly from the developments of $(a+b)^n$, we deduce the formula for

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 -$$

&c.

27. Numberless illustrations may be given of the application of the rule. We shall content ourselves with mentioning two more general formulæ: the first pertaining to simple quadratic equations, the other to simultaneous equations of the first degree with two unknown quantities.

From the general equation of the 2d degree

$$x^2 + px + q = o,$$

we deduce

$$x = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}$$

Now if we have given the equation

$$x^2-x-20=0,$$

on comparing it with the first one we note that

$$p=(-1) \text{ and } q=(-20),$$

hence, substituting these values in the expression for x, we find the value in this case.

$$\therefore x=-\frac{(-1)}{2}\pm\frac{1}{2}\sqrt{(-1)^2-4.(-20)}$$

$$\therefore x=\tfrac{1}{2}\pm\tfrac{9}{2}=5 \text{ or } -4.$$

Again take the equations

$$\left.\begin{aligned} ax+by&=c \\ a'x+b'y&=c' \end{aligned}\right\}$$

By elimination we obtain

$$x=\frac{cb'-bc'}{ab'-ba'}$$

$$y=\frac{ac'-ca'}{ab'-ba'}$$

In case any of the numbers a, b, c, a', b', c' are minus, in any particular example, we can at once write the values of x and y from the last two equations by a simple substitution, without the necessity of an actual elimination process.

This is evident, because in this process in the general case we should have only to substitute $(-a)$ for a, etc., throughout, to get the corresponding expressions where a, . . . were minus.

We have now deduced all the fundamental rules of algebra from a few comprehensive definitions relating to the meaning of + and − signs in connection with numbers, combined in various ways, and have shown the wonderful generality given to formulæ by the use of the algebraic negative quantity.

28. We shall now discuss some problems where we shall similarly reach the greatest generality by the same means. In fact the use of negative quantities is necessary in certain problems, where although $(a-b)$, to take an illustration, may be assumed as plus at the beginning, yet it may turn out in the end that $a<b$, so that we have supposed an impossible subtraction if a and b are regarded as absolute numbers. By the use of the algebraic series, however, all difficulty disappears, not only in the work but also in

the interpretation of the answer when minus; though here it is necessary to state in the beginning the positive sense, when the negative will be just the reverse.

29. *Problem* 1.—A's age is now a years, and B's age is now b years; how many years before or after this epoch to the period when A's age is m times B's.

Let x=number of years, regarded as + if reckoned in the future, − if in the past; then in any case

$$a+x=m(b+x).$$

We see that if x turns out to be plus, the formula is correctly stated, as well as when x is minus, or when the period antedates the present.

Solving we at once find

$$x=\frac{a-mb}{m-1}$$

Now, if $a=40$, $b=15$, $m=2$, then $x=+10$ years; *i. e.*, in 10 years A's age will be double B's.

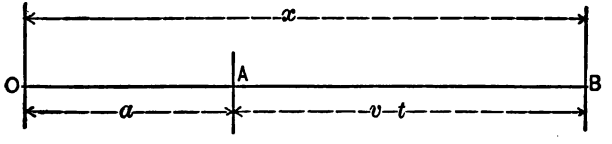
But if $a=35$, $b=20$, $m=2$, then $x=-5$, that is, A was twice as old as B 5

years ago. It is not necessary to make a new statement of the problem, so that x shall come out +, as is generally done. The result is perfectly intelligible as it stands.

30. *Problem* 2.—The celebrated problem of the couriers. Two couriers, at a certain instant of time, are at certain points on a straight road, and moving towards or from each other at certain uniform velocities. When were they together, or when will they be together?

It seems impossible, at first glance, that one solution can cover all the cases that can be imagined, but we shall find it to be so.

In the figure below the first courier is supposed at the point A, distant a from O, *the same instant* the second courier is at some other point A′ (not shown in the figure), distant a' from O.



Now, the two couriers are supposed to meet at B, distant x from O, either t hours before or t hours after passing the two points A and A′; in the first case t will be called minus, in the second case plus. Similarly,

x, a and a' will be + if measured to the right of O, otherwise −.

The rate per hour, or velocities v and v', at which the couriers travel, will be called + if the motion is to the right, − if to the left in either case.

We are thus carrying out principles already employed in article 11, that distance to the right will be called +, if distance to the left is minus; time after is +, if time before is −, and motion to the right + when motion to the left is called minus.

Where all the quantities are plus, we can at once write the formula for the first courier,

$$x=a+vt \quad \ldots\ldots \quad (1).$$

But, by art. 11, this formula holds in all cases, for $x=a+vt$ is the algebraic sum of two numbers, and thus gives the

distance of the courier from the origin O at the time of meeting, this distance being measured to the left if x is minus, otherwise to the right.

This can be made plain by noting that if v is + and t +, the product vt=distance AB is + ; if the courier is going to right still, but he met the other one t hours before reaching A, then t is —, and vt is —, as it should be, since B is to the left of A.

Now, for the two cases where the motion is to the left, or v minus, t may be + or — ; if t is +, or the time of meeting was t hours after the courier passed A going to the left, then the term vt will be minus in the formula; when t is — also, vt will be plus, for now the point of meeting is to the right of A.

The reader can readily illustrate the various cases, dependent upon the signs of a, v and t, by diagrams. For the second courier, we should have the analogous formula for the distance OB,

$$x = a' + v't \quad \ldots \quad (2),$$

whence, placing the two values of x equal, we deduce,

$$t=\frac{a'-a}{v-v'} \quad \ldots \quad (3),$$

$$x=\frac{va'-av'}{v-v'} \quad \ldots \quad (4).$$

At the same instant of time the two couriers are respectively at the distances a and a' from O, so that $(a'-a)$ in (3) represents their distance apart at that instant (article 13). Likewise $(v-v')$ is the arithmetical difference or sum of their velocities according as they are going in the same or opposite directions; hence we know eq. (3) to be correct independently of previous considerations.

As a particular case, let $a'=10$ miles, $a=(-20)$ miles, $v=4$, and $v'=6$ miles per hour, then from (3) we find,

$$t=\frac{10-(-20)}{4-6}=\frac{30}{-2}=-15.$$

That is, they met 15 hours before the time first courier was at A, distant 20 miles to the left of O.

Now, in 15 hours the first courier

travels $4\times15=60$ miles, so that they met $60+20$ or 80 miles to left of O, which we likewise find from formula (4),

$$\therefore\ x=\frac{4\times10-(-20)6}{-2}=-80.$$

For the special cases when $v=v'$, (3) gives,

$$t=\frac{a'-a}{o}=\infty;$$

or they will never meet, as is evident, since the assumption is, they have the same velocity and are going in the same direction.

If, besides, $a'-a=o$, or they are together at the start, they will be together at all times, so that t can be anything. Here the equation takes the singular form,

$$t=\frac{o}{o}$$

31. We have now given the theory and a sufficient number of illustrations of the use of the algebra of algebraic numbers to show the generality obtained by its use. The theory has been developed by giving comprehensive meanings to the

signs + and −, so as to include arithmetical algebra as a special case when the numbers or symbols representing them are all plus quantities.

It is not proposed to develop the theory further, except to note an extension of meaning given to "greater than" or "less than," with a necessary caution.

Thus an algebraic quantity is said to be *greater than* another when it lies in the positive direction from it.

Thus, we say that $5 > -1$, and $-2 > -3$. This is a very useful convention in generalizing results, but it must not be inferred from it that there is such a thing as a *ratio* between a + and a − quantity.

Before the algebraic series of numbers, with the use to be made of it, had been brought forward, there was considerable difficulty in the comprehension of negative quantities as they appeared in analytical investigations, especially where a minus quantity was defined or understood, in some fashion, to be less than zero.

Thus Carnot follows D'Alembert in quashing the whole business as follows:

Consider the proportion.

$$1 : -1 :: -1 : 1$$

If -1, is less than zero, then we have a greater is to a less as a less is to a greater, which is an absurdity ; and yet the product of the extremes is equal to the product of the means, so that the four quantities should be in proportion apparently. Q. E. D.

But we have tacitly assumed here that there is a ratio of 1 to -1, and it is just here that the absurdity is introduced, for from the original definitions of $\times$ and $-$ quantities we see that they are of such opposite character that they can have no ratio, though by the technical rule for division, (which is deduced from that for multiplication) the quotient $\frac{-}{+} = (\frac{+}{+}$ and the direction reversed) giving -1 for the result, is the same as $-\frac{+}{+}$.

If we choose to *extend* the ordinary arithmetical meaning of *ratio* to include algebraic numbers, well and good; but

with such merely conventional definitions, it would be next in order to *ascertain*, if in any proportion, we must always have,

greater : less : : greater : less.

And we should soon find, by the consideration of this very example in fact, that such a criterion does not exist.

32. This example should serve as a valuable caution to the student, against assuming anything that may turn out to be inconsistent with the definitions. The algebraic series of numbers is essentially much more comprehensive, in most respects, than the natural series, so that in using it as the basis of an algebraic system, we should be all the more careful to give few and comprehensive definitions that are consistent throughout, and deduce the laws of the new algebra entirely from these definitions and obvious elementary principles.

I have fault to find with writers of text books in this very respect, that a number of definitions or rules pertaining to the fundamental operations are given,

which to the student seem not to have the proper connection, or in fact may be conflicting; so that he cannot give his full assent, or at least reserves his judgment until the whole ground can be carefully reviewed and the bearing of each case upon every other closely analyzed.

We have endeavored, in what precedes, to deduce the laws from obvious principles, after giving precise, though comprehensive definitions of the meaning of + and − whenever used, whether in multiplication or division, addition or subtraction.

There must be but one meaning to each of these signs, and that meaning is very simple: plus meaning counting forwards, or in the usual direction; minus counting backwards or in the reverse direction. And yet from these simple definitions, we devise the algebraic series of numbers; and then deduce, with definitions consistent with the first, all the fundamental ground rules of addition, subtraction, multiplication and division

for both simple and compound expressions.

Thus we have not to give new definitions of + and — under the various heads enumerated, and perhaps leave the student to find out if these meanings be consistent, but we agree in the first instance that wherever + or — occur, we shall understand but one meaning to each of them. Then there can be no inconsistency; for a sign, as —, before either a single term, a product, a quotient or any compound expression, means simply but one thing, that the indicated result must be counted in the reverse direction to that assumed for the positive direction.

Again, supposing the ground rules established in a logical manner, illustrations should follow, without which the student of algebra often wonders why isolated negative quantities are introduced at all; and then again whilst the principle of substitution in a general formula, as illustrated in previous articles, is universally made use of, yet I am un-

acquainted with any attempt at demonstrating the universal correctness of the process.

The truth is the books give too little, and that in too disconnected a form, for the student unaided, to grasp the entire subject. Especially is this true in the transition from multiplication of simple to that of compound expressions.

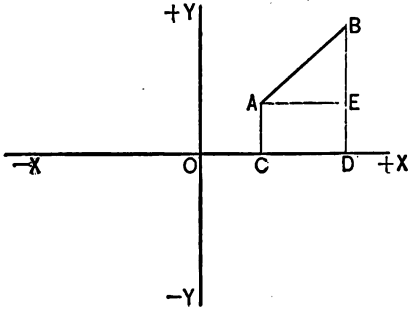
33. It would lead us far beyond the scope of this article to take up the numberless applications of the algebraic numbers to trigonometry and the higher analysis. Besides in trigonometry, modern text books have already shown the generality of all the formulae by the use of negative numbers, so that little remains to be said. In analytical geometry, though, there is a singular lack in demonstrations going to show the universality of the formulae deduced, to all angles and quadrants; and yet this very generality is often assumed, as if either, sufficiently evident, or following from some sort of general principle previously demonstrated. Now, neither of these

suppositions is correct. No student finds the generality assumed, either self-evident or following from principles already demonstrated. In fact, it is absolutely necessary to show, *for each case*, that the formulae deduced is general, for problems can easily be given, in the application of algebra to geometry, where this generality cannot be attained. This is an important difference between purely algebraic formulae and those used in the solution of some geometrical problem. Further, it will often happen that a formulae deduced for a certain case, may include other cases of a different kind, where the algebraical transformations have been such as to increase the number of the roots, over those originally contemplated. Thus in squaring an equation and then reducing and solving, we can introduce double the number of roots intended originally; but generally the false solutions are easily detected.

34. It will be found, too, as a rule, that the generality of the formulæ of analytical geometry and trigonometry rests on

the tacit assumption that the rules of signs, as laid down in algebra, are to be observed, although there may be no meaning attached to some of the operations in themselves in certain cases. To make my meaning clear to the students of analytical geometry, let us find the analytical expression for the distance between two points whose co-ordinates are respectively x_1, y_1, and x_2, y_2.

Let A and B in the figure, represent the points, so that drawing AC and BD parallel to OY, we have $OC=x_1$, $AC=y_1$; $OD=x_2$, $BD=y_2$.



Then, by the theorem that "the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides," we have,

$$\overline{AB}^2=(x_2-x_1)^2+(y_2-y_1)^2.$$

To ascertain if this formula is general or true, no matter what the co-ordinates x and y, let us first suppose y_1 and y_2 to remain constant, so that A and B will both be above the axis of X, and remain at the same distance from it. Then if B lies to the left of A in the first quadrant, (x_2-x_1) will be negative, but its square $(x_2^2-2x_2x_1+x_1^2)$ will be the same as before, so that the formula is unchanged.

Again, if A and B are both to the left of OY, then x_1 and x_2 are both minus, but $(x_2-x_1)^2$ is equivalent to the square of the projection of AB on OX, as in the former case. Similarly, when A and B are on different sides of OY; for now (x_2-x_1) will no longer be an arithmetical difference, but a sum, or minus a sum, which is the projection of AB on OX, so that $(x_2-x_1)^2$ will be equal to the square of

this projection, as before. It is, moreover, evident that the term $(x_2-x_1)^2$ will remain of the same form whether A and B are both above, both below, or one above and the other below OX.

We can prove in an exactly similar manner that $\overline{BE}^2=(y_2-y_1)^2$, no matter in what quadrants A and B may be found, or what the relative values of y_2 and y_1. So that the *one* formula is true for every supposable case, with the understanding, implied as it is, that the rules of signs as deduced in algebra are in force here, and of course the signs of x and y as given in analytical geometry. Thus, to take a numerical example, let

$$x_2=-5,\ x_1=+2,\ y_2=+4,\ y_1=+6,$$

then,

$$\begin{aligned}\overline{AB}^2&=(-5-(+2))^2+(4-6)^2\\&=(-7)^2+(-2)^2,\\&=49+4=53,\end{aligned}$$

$$\therefore\quad AB=\sqrt{53}.$$

Now, the point that I have to make is this, that the formula in the form,

$$\overline{AB}^2=(-7)^2+(-2)^2$$

means nothing, though it leads to a true formula,

$$AB^2=(7)^2+(2)^2,$$

which carries out the theorem that the square on the hypotenuse equals the sum of the squares on the other two sides; whereas $(-7)^2$ cannot be said to represent the square on a side, for $-7\times-7=(-7)^2$ conveys no such meaning as the square on a side. Hence, although the general formula, *as it stands*, does not seem to carry out the principle of the square on the hypotenuse, &c., for *any* values of the co-ordinates, yet for any such values at pleasure, it can be at once transformed into a formula which is an analytical expression of this theorem, so that the formula is said to be general.

We note, further, that on extracting the square root, in the example above, according to the laws of signs, that we get two solutions,

$$AB=\pm\sqrt{53};$$

but the minus result must be discarded, since AB was tacitly assumed as positive from the beginning. This illustrates another point noted above.

NOTES ON GEOMETRY,

WITH

ESPECIAL REFERENCE TO THE METHODS EMPLOYED.

GEOMETRY is justly considered one of the most exact of sciences; since the axioms and definitions that form its bases are regarded as such that every one is willing to admit, whilst the propositions are just deductions from the true assumptions.

It is evident, however, that if any of the hypotheses can be simplified, there will be a real gain in the perfecting of the science, which should be founded on the fewest number of things regarded as self-evident, especially if there should be the slightest suspicion that any of them may be contradictory. Again as to the methods of reasoning employed in deducing the various propositions from

the given data, there is evidently choice, though modern authors have confined themselves so strictly to the deductive method, that the inductive or ancient analysis has scarcely received a passing notice from them. As the latter method is of such great value in scientific discovery, its complete exposition as a perfect method of reasoning, when rightly understood, is very important, and is well worthy of the most attentive consideration.

In developing the subjects under consideration, a brief exposition of general methods, applicable to all sciences of reasoning, is first given as naturally preceding the critical notes and applications to geometry.

Whatever of value is found in the following pages is due mainly to Duhamel, whose "Méthodes dans les Sciences de Raisonnement" has been freely used in the preparation; and it is hoped that none of the acute, searching thoroughness, so characteristic of that author, has been lost in the transfer.

Methods Common to all Sciences of Reasoning.

1. Necessary truths exist by themselves; reasoning and methods are only means that man employs to discover them or to recognize them, and their object is to produce in him the knowledge and *certainty* of the truths.

This state of certainty is caused in man by the clear conviction of the truth; in other words, as Descartes long ago observed, by the *evidence* or proof of it.

Truths whose obviousness occurs to all minds are used as the starting point, and the *methods* serve to discover other truths which can then be admitted with the same certainty as the first.

2. The ordinary form of the *syllogism* is a form of reasoning in which the characteristics belonging to a group are simply reaffirmed with respect to the individuals of that group, which is so simple that it scarcely needs giving a name to. It does not help to discover anything, for we must know or ascertain the qualities of the individual before we can af-

firm them for the group, and therefore it will not be included in the general exposition of methods.

3. The errors that we commit in reasoning proceed less often from a vicious deduction than from the inexactness of the propositions admitted, which are often, not those called *axioms*, which every one would admit from experience to be true, but rather plausible hypotheses, not so evident, which may be true for special cases, but are not so in all their generality, which, however, is often assumed as evident.

The only way to escape error in this direction is to examine carefully and perhaps tediously, every case that the statement involves, and verify it in each instance, or state the exceptions where the statement is not general, before proceeding with the investigation.

4. The operation of deducing a result from known results is called *deduction;* it corresponds to *synthesis*, and is the method generally used in communicating

truths, that have been previously ascertained, to others.

5. The method of *induction* is the operation by which we refer the knowledge of a thing to that of other things of which it will be a consequence.

This corresponds to the *ancient analysis*, and is generally the march of discovery. It will therefore be fully developed as we proceed.

6. Propositions that cannot be true at the same time are said to be *incompatible*. Thus, to say that x is greater than y is incompatible with the statement that y is greater than x; but both statements can be false at the same time, which would happen if x were equal to y.

But if each proposition is the negative of the other, or *contradictory* to it, if one is true, the other, which denies it, will be false; and, if one is false, the other, which only denies it, will be true. We must be careful, though, not to omit any of the cases of the contradictory proposition, otherwise we should have a proposition that was simply incompatible

with the first. Thus, if we assert that $x>y$, the contradictory proposition would assert that x was not greater than y, which includes two cases, viz., $x<y$, and $x=y$. If we simply said that $x<y$, the two propositions would be incompatible, but not necessarily contradictory, for both would be false if x equals y, so that we could not, if we recognized the falsity of one, thereby infer the truth of the other.

7. It must be carefully borne in mind, also, that truth may sometimes be deduced from falsehood, so that it does not follow because a deduction is true that the propositions from which it is deduced are true. They may be, one or more, false.

Thus, Aristotle gives the following odd illustrations, in the first of which the two premises are false, and the conclusion true:

Man is stone;
All stone is animal;
Then mankind is animal.

In the following, one premise only is false and the conclusion true:

All horses are animal;
Man is not animal;
Therefore, a man is not a horse.

Many illustrations could be given, but it suffices to remark that we may reach a true result from a false principle, either because the latter is a mixture of true and false, and we employ only the true, or that the errors made in the deductions compensate each other; thus it is proved that if we reach a true result, it does not follow that the relations admitted in deducing it are true.

8. It is equally important to observe, that if in going from certain relations we deduce, by just reasoning, a false conclusion, that *the first relations are not all true.*

For if they were we could only deduce true conclusions.

9. The science of a thing is the whole of the laws of that thing, so that the science of reasoning involves the laws of reasoning.

10. A *theorem* is a proposition to be demonstrated; a *problem*, something to be done, satisfying given conditions.

In a problem we indicate the result we wish to obtain and demand the means of attaining it. A theorem is true or false, a problem possible or impossible.

11. *Analysis.* The "*ancient analysis*," which alone we shall consider in this paper, does not seem to be well known by modern logicians. We find the first traces of it in the elements of Euclid, though Pappus of Alexandria refers the invention to Plato.

The method can be applied both to theorems and problems, which applications will be considered separately.

12. *Analytical methods for the demonstration of theorems.*

When we have to find the demonstration of a given proposition we shall seek first if it can be deduced as a necessary consequence of admitted propositions, in which case it should be admitted; it will thus be demonstrated.

If we cannot perceive from what known

propositions it can *be deduced*, we shall seek from what proposition not admitted it can be, and then the question will be to demonstrate the truth of the last. If this *can be deduced* from admitted propositions it will be true, and consequently the proposed will be; otherwise we shall seek from what proposition not admitted it *can be deduced*, and we then have to prove the truth of the last, if possible.

We shall continue thus until we reach a proposition that may be recognized as true, and then the truth of the proposed will be demonstrated.

We see, therefore, that this method that we call analysis consists in establishing a chain of propositions, commencing with the one we wish to demonstrate and ending with a known proposition, so that in going from the first, each may be a necessary consequence of that which follows; whence it follows that the first is a consequence of the last, and therefore as true as it is.

13. *Definition.* When two propositions are so related that either one can be de-

duced from the other as a necessary consequence they are said to be *reciprocal.*

14. Now, if any two successive propositions, found after the general method above are reciprocal, we can consider the second as deduced from the first, and *if this reciprocity holds from the first to the last,* we can say that the analytical method consists in establishing a series of propositions of which the first is the one to be demonstrated, and such that the second is deduced from the first, the third from the second, and so on to the last proposition which is recognized as true.

15. It is by no means necessary that all the successive propositions be reciprocal; only if some are not so then the general method of art. 12 must be applied to such propositions.

If we employ the second manner of proceeding (art. 14), which is often easiest (as it is generally simpler to deduce a consequence from a proposition than to find another of which this will be a consequence), and do not ascertain that the successive propositions are reciprocal,

if for only two, say; the second of the two can be true without the first being true, *since the truth can sometimes be deduced from error* (art. 7); so that we can not say that the first proposition is demonstrated.

Duhamel is the first author who has insisted on this reciprocity test, and it is well to call especial attention to it, for its neglect may lead to grave errors. Of course, where it does not obtain, the general method of art. 12 must be used.

16. *The analytical method for the solution of problems.*

The object of a problem is to determine one or more things of given kinds, so as to satisfy given conditions. If we can refer the problem, then, to another whose conditions satisfy the first, so that the knowledge of the second comprehends at least some solutions of the first; if the new problem is easier to solve than the first, the question will be advanced.

If the second cannot be resolved, refer it to another as the first was referred to it, and so on until we reach a problem we

can resolve. Its solution will furnish one or more solutions of the proposed.

17. Now, in this general method the successive problems have been so chosen that the conditions of any problem satisfy those of the preceding. Now, if the conditions of any one satisfy as well those of the succeeding one, the two conditions are said to be *reciprocal.*

If this does not obtain, the general method may not find all the solutions, so that there may be *lost* solutions in this case; for calling any problem (1), and the succeeding one, whose conditions satisfy those of the former (2), it is plain that a solution of (2) will give some of the solutions of (1) for the same conditions or requirements that satisfy (2) satisfy one; but some solutions of (1) may not be solutions of (2), for, by assumption, all the conditions of (1) do not necessarily satisf y (2),so that there will be more solutions of (1) than we should find by solving (2). But if, on the contrary, the conditions of the second problem and those of the first are reciprocal

when the first is satisfied the second is also, therefore all the solutions of the first problem will be solutions of the second, and *vice-versa*; so that if we find all those of the second, none will be lost.

Reasoning in the same way for the others, we can state generally the following proposition:

If in the problems that we substitute for the preceding, in going from the proposed, the conditions of any two consecutive ones whatsoever are reciprocal, the solutions of the first involve those of the last, and of any one whatsoever of the others, and reciprocally.

In the above reasoning we have not required that the things demanded be the object of research in each problem; but if they are different, we call the conditions of the two problems reciprocal, when knowing the things satisfying the imposed relations in either one of the problems, the corresponding things of the other satisfy the conditions imposed upon them in this other; in other words, when the complete solution of either prob-

lem entails the complete solution of the other.

18. It may conduce to clearness to mention the successive steps in the analytical solution of a problem given by Duhamel in his second book, which is too long to reproduce entire.

PROBLEM—*To draw a circle tangent to three given circles.*

It is easily shown that the solution of this problem involves that of this problem, *to describe a circle which passes through a given point and is tangent to two given circles* and *vice versa.*

The two problems thus have reciprocal conditions, for the things satisfying the conditions of the one require as a necessary consequence that the corresponding things satisfy the conditions of the other.

We have thus referred the given problem to another whose solution embraces that of the first.

Similarly we refer the second problem to a third;

To describe a circle which passes

through two given points and is tangent to a given circle,
whose solution involves that of the second, the conditions being reciprocal; when again, we refer the last to the problem,

To pass a circle through three points, and with reciprocity.

It is to be understood that the constructions effected in referring the first to the succeeding problems have all been made, so that when the centre of the circle corresponding to the last problem is found (which we know how to do), this is none other than the centre of the required circle, which is thus easily drawn tangent to the three circles, and the original problem is solved.

19. Let us next take the case, where the conditions of the new problem substituted for the first are simply consequences of those of the first, but not reciprocal to them.

It follows that any solution of the first involves the solution of the second, since all that satisfies the conditions of the first,

necessarily satisfies the conditions of the second; but *all* the solutions of the second are not necessarily solutions of the first, since the hypothesis is that the conditions are not reciprocal, and it will therefore be possible to satisfy the conditions of the second without satisfying those of the first. The second problem in this case has, then, *all the solutions* of the first, but it may contain *strange* solutions in addition.

20. Remark.—The writer is constrained to think that the reasoning of Duhamel in the preceding article is obscure and possibly requires elucidation. First, it seems to be tacitly assumed that the second problem substituted for the first is not simply *a particular case* of the first, for then the solution of the second might not include *all* the cases of the first, but that the conditions of the second are just as comprehensive as those of the first and include every case of the first, so that the solution of the second gives every solution of the first. But then it may be asked, why should it contain more, if it is

simply a consequence of the first? How can a consequence, if it is simply a consequence and nothing more, include more than the original proposition?

The truth is, that in the examples illustrative of "strange solutions," it will be found that the so-called consequence, is a consequence only for certain values of the quantities entering it; whereas, in solving it, we tacitly assume it to be true for all values of its quantities, so that we find strange solutions to the original problem.

In fact the assumption, in art. 19, that the old and new conditions are not reciprocal, implies that the new conditions are more comprehensive than the first, and that only part of them are direct consequences of the first; for if all were, there would necessarily be reciprocity. For these reasons, I prefer to put the deductions of article 19 in the following form:

When the conditions of the old and new problems are such, that *all that satisfies the first satisfies the second, but not reciprocally*; then the complete solution

of the second problem will certainly give all the solutions of the first, with perhaps *strange solutions*, if the conditions of the second problem are more general than those of the first and not all strictly consequences of the first. The solution of the following problem will elucidate what has been said:

21. PROBLEM.—*Knowing the hypothenuse of a right triangle, the sum of the other two sides and the altitude, to compute the last three quantities.* Call a this sum, b the hypothenuse regarded as the base; x and y the two sides, and z the altitude. The known properties of the triangle and the relation imposed in the enunciation give the three following equations:

$$x+y+z=a \quad . \quad . \quad . \quad . \quad (1)$$
$$xy=bz \quad . \quad . \quad . \quad (2)$$
$$x^2+y^2=b^2 \quad . \quad . \quad . \quad . \quad (3)$$

Thus we refer the geometrical problem to one of numbers. To solve the latter, transpose the z in eq. (1), and squaring, we have,

$$x^2+2xy+y^2=(a-z)^2 \quad . \quad . \quad . \quad (4)$$

subtracting eq. (3) from this, and double eq. (2) from this result we obtain, on reducing,

$$z^2-2(a+b)z+a^2-b^2=o \quad . \quad (5)$$

whence,

$$z=a+b\pm\sqrt{2ab+2b^2} \quad . \quad . \quad (6)$$

But it would evidently be wrong to take the solution that corresponds to the + sign of the radical, for z, the altitude of the triangle cannot be greater than $a+b$; whence substituting the other value in (1) and (2), we find the values of x and y by elimination between the first three equations.

But, we can ask, why have we found two values for z when one only applies, the other being "strange" to the solution. Duhamel says, "now we have truly demonstrated, that eqs. (1), (2) and (3) were *consequences* of the geometrical conditions, but not the reciprocity, whence the strange solutions." The truth is (1), (2) and (3) are not *simply* consequences of

the geometrical conditions, but enclose in themselves other solutions, *for if x and y are both written with the minus sign*, we shall obtain the same value of z, eq. (6) just found. This is plain, for eq. (4) can equally be written $(x+y)^2=(a-z)^2$, or $(x+y)^2=(z-a)^2$ and the results of the calculation will be the same. Therefore, eqs. (1), (2) and (3) satisfy not only the geometrical conditions of the problem, but likewise another problem of numbers, in which x and y are both negative at the same time.

22. The conclusions arrived at concerning the solution of problems may then be summed up as follows:

1° If the conditions of the successive problems to which we refer the proposed are reciprocal, in order, beginning with the proposed, the solutions involve each, the others.

If the successive conditions, in order, are not reciprocal, then:

2° If some of the conditions relative to any one whatsoever of these problems are simply consequences of those of the

following, all the solutions of any one whatsoever are solutions of the proposed, but may not include them all.

3° If some of the conditions of any one are consequences of the *preceding*, so that what satisfies the latter will satisfy the former, the solutions of any one whatsoever comprehend all those of the proposed, and may include strange solutions.

If several of these cases hold successively in the solution of the problem we apply the proper principles in turn to each case.

23. THE SYNTHETICAL METHOD.—By this method, for the demonstration of theorems, we start with known propositions and deduce the others successively as consequences until we arrive at the theorems proposed, which is thus recognized as true.

Similarly in the resolutions of problems we start with a known solution and deduce a new one from it, a new one from this and so on, until we reach the proposed, whose solution is thus obtained.

24. The synthetical method is generally

used in communicating results to others, and often in discovery, whether deducing consequences from known results at random, or by intelligently combining known results with a view to a certain end. As it is often difficult to know from what known problem or theorem to start, we can often make a series of fruitless essays before hitting upon the proper solution.

The analytical method possesses the advantage of a starting point, as we know from what result we have to go to seek one from which it is to be deduced; and this is often a great advantage, even in communicating results; for by the synthetical method, the reader often sees no object in the various steps of the demonstration or solution, until the end is attained.

No rules can be given as to the best method to employ—sometimes both may be employed in the same investigation—for by *deducing* certain results having more or less analogy to the proposed, we stand some chance of applying them in the course of the analytical march.

25. The ancient authors did not perfect the analytical method. Thus Euclid and Pappus both regarded a proposition as demonstrated when they could deduce from it a proposition known to be true. The examples solved by them carry out this idea, which is evidently defective, since we have seen that false premises can sometimes lead to true conclusions. After reaching the true proposition, if they had then reversed the order and verified the preceding results synthetically, the results would be proved true; or still better, if they had shown the reciprocity in the successive theorems or in the conditions of successive problems, the results would be true without the necessity of a synthetical verification. It is little wonder that the analytical method should be treated with so little favor, when its limitations and requirements were so imperfectly perceived.

26. How to demonstrate that a proposition is false.—If we can deduce by just reasoning from a proposition, temporarily admitted, a false proposition, or

one *incompatible* with any of the preceding, the assumed proposition is false; for if the first was true, all those deduced from it would necessarily be true, and could not consequently offer any incompatibility with it or any of the others.

This method is analytical, in that it takes for a point of departure the proposition in question and goes to some desired result, whose truth or falsehood thus establishes that of the first proposition.

27. Reductio ad absurdum.—Two propositions, of which each is the simple negative of the other, are contradictory (art. 6), and consequently, if one must be true, the truth of one involves the falsity of the other and reciprocally; so that if we can prove that the contradictory proposition to one is false, that one will be demonstrated true, provided that every contradictory case is stated. This method, called the "reductio ad absurdum," simply amounts to this:

Let A, B and C represent *all* the possible cases of a certain proposition, *one*

of which must be true. Now if we can prove that cases B and C are false (art. 26), then it must follow logically, that A is true, for one must be true, and case A is the only one not discarded as false. The latter method of putting the case is seized much more readily by students than the former, and amounts to the same thing as B and C are here contradictory to A, so that the first demonstration applies with the same results in both cases.

We have now given a rapid resumé of the laws of reasoning, and will now proceed to their application to geometry.

Application of General Methods to Geometry.

28. We are met at the threshold of the science of geometry with difficulties relative to proper definitions of the straight line and plane.

Euclid defines a straight line as one which rests evenly on its points, and adds, as an axiom, that two straight lines cannot enclose a space. The definition gives but little meaning, and should be rejected.

Modern authors define a straight line as the shortest distance between two points, and assume that any two straight lines can be made to coincide, besides often giving as an axiom a third property, that a straight line has the same direction throughout its whole extent, without inquiring whether some of these requirements may not be incompatible. Duhamel evidently takes a hint from Euclid's axiom, and defines *a straight line as such that through any two points we can only pass one straight line.* It follows that such a line will coincide with any other straight line when it has two points in common, and that the form of the line is everywhere the same.

This is a precise definition, but it does not immediately give us an idea of the form of the line meant by it.

If we define a straight line as one having everywhere the same direction, we form an idea at once of the kind of line meant, though the word direction must then be defined, as it is often associated

with the path of a certain course on the earth's surface. It would be more precise to define *a straight line as one every point of which is in line with any two points of the line assumed at pleasure.* We sight along a straight edge to see if it is "straight," or satisfies the above condition, and this is really the first and most lasting impression we get of a straight line. Something has to be assumed as the result of observation or experience, no matter what definition is used. If we employ the last definition then it follows as a consequence that but one straight line can be drawn between any two points, which thus includes Euclid's axiom as a consequence, as well as Duhamel's definition, and the further fact that two indefinite straight lines coincide throughout their whole extent when they have two points in common. Accepting one of these definitions it should be next in order to *prove* that a straight line *is* the shortest distance between two points, and not assume it, as is done by the followers of Legendre.

We shall enter into this demonstration directly, after defining a plane and showing some of its characteristics.

29. The usual definition given of a plane—such a surface that a straight line connecting any two of its points will lie wholly within or on the surface—is certainly the one derived from experience, for on looking over a smooth sheet of water we notice that it looks straight from point to point. Any one trying a straight edge on a blackboard will rest satisfied that there is such a surface, so it may be pertinently asked why not accept the definition, since we know that there is just such a surface as is described by it.

There can be no objection, that I see, from a practical point of view, but it is very desirable in pure theory to assume as little as possible, especially if the thing can be deduced from previous assumptions and axioms.

Therefore Duhamel conceives a plane to be generated by a straight line revolving about a point in another straight

line and remaining always perpendicular to it. It is very evident that such a surface can exist, and we have only to show that a straight line passing through any two points of it lies wholly within the surface.

To prove this, conceive the surface completely turned over, so that the fixed straight line shall be in continuation of its first position, then if the point, the intersection of the perpendiculars to the fixed line, is the same for the two surfaces they must necessarily coincide, since both are the loci of perpendiculars to the same fixed line at the same point. [For the same reason, if we revolve either surface about the fixed line it will coincide throughout with the other surface, though this fact is not necessary to the demonstration below.]

Now pass a straight line through *any* two points of the common surface, it will coincide with the surface throughout its whole extent. If you deny it, then it must leave the surface somewhere, either on the one or the other side; but any

reason that may be given for its leaving the first surface on the side of the fixed line or the reverse, will apply equally well to the second surface, so that the line would have to be on both sides of the common surface at the same time, since the fixed line (or directrix) for one surface is on the opposite side of the common surface to that for the other surface.

Hence, as the straight line drawn through any two points of the plane must either lie in it entirely *or not*, and we have found the latter supposition absurd, the fact must hold (article 27).

It must follow, likewise, that any indefinite straight line which passes through any point outside of this locus (the plane surface), can have not more than one point in common with it, for if it had two, all its points would coincide with it, which is contrary to the hypothesis.

30. From the property just proved of the plane we can easily show that any two planes having any three common points, not in the same straight line, will coincide

throughout. For, draw a straight line through two of the points, it will coincide with both surfaces throughout, and if we conceive an indefinite straight line to revolve about the third common point, always touching the line drawn, it will coincide with *both* surfaces at the same time in all its positions, since it has two points in common with them; therefore the two planes present a common surface, and coincide throughout.

31. If preferred, a plane may be defined as a surface generated by revolving a straight line about a fixed point, and always touching a fixed straight line not passing through the point. For, reversing the surface, and reasoning as before (29), we prove that a straight line will coincide with it from point to point, whence the conclusion of art. 30 holds.

32. We have given the reasons why the definition, that "the straight line is the shortest between two points should not be accepted. It is a thing to be proved in due course, and we shall now give the few theorems, necessary to prove

it and which should form some of the first theorems of any treatise. The demonstrations are taken mainly from Euclid, who did not fall into the error of modern geometers of assuming the principle in question when it was susceptible of proof. As the reader is supposed to be versed in geometric processes the demonstrations will be made very brief.

33. Using Duhamel's definition, that a straight line is such that but one straight line can be drawn through two points; or if preferred, the other that I suggest, that a straight line is such, that every point of it is in line with any two points of it whatsoever, which includes the above definition, we easily show in the usual manner, that,

If two straight lines cross each other, the vertical or opposite angles are equal.

Also, by superposition, it is shown in the usual manner that,

Two triangles are equal when they have an equal angle comprehended between two sides equal each to each.

34. Next we shall show, that,

The exterior angle of a triangle is greater than either of the opposite interior angles.

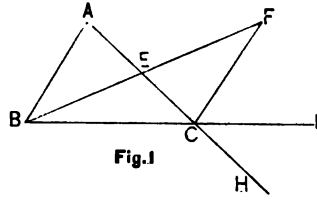


Fig.1

Let A B C (Fig.1) be a triangle, and ACD the exterior angle that we wish to prove greater than either BAC or ABC. Take E the middle of AC, draw BE and extend it to F, a distance EF = BE. Then the triangle AEB is equal to the triangle CEF (article 33) having the opposite angles at E and including sides, equal each to each. Therefore the angles A and ACF are equal; but as ACF is less than ACD, from the construction, it follows that angle ACD is greater than the opposite interior A.

In a similar manner, by bisecting BC, &c., we can prove that the exterior

angle BCH is greater than ABC; therefore, since BCH = ACD, (33), we conclude that any exterior angle is greater than either of the opposite interior, Q.E.D.

35. We shall next show that,

In an isosceles triangle, the angles opposite the equal sides are equal.

In the triangle ABC, let AB = AC (Fig. 2),

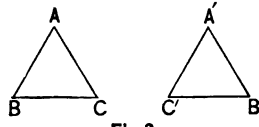


Fig.2

Turn the triangle ABC over, and suppose it to take the position A′B′C′, the angle B falling at B′, C at C′. Now slide the second triangle so as to coincide with ABC, which it can be made to do completely, since the angles A and A′ are equal, side AB = A′C′ and AC = A′B′. Therefore, the angles B and C′ = C, are equal, which was to be proved.

36. *If two sides of a triangle are un-*

equal the angle opposite the greater will be the greater.

Let AC>AB (Fig. 3); take AD=AB, the triangle ABD will be isosceles, and

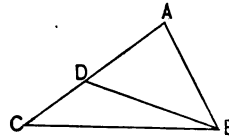


Fig.3

consequently, the angles ABD and ADB will be equal (35). But the angle ABC> ABD=ADB; and ADB>C (34): then for a stronger reason, ABC>C, which was to be proved.

37. *If two angles of a triangle are unequal the greater side will be opposite the greater angle.*

There are only three possible cases and one of them must be true; either the side opposite the greater angle must be less than, equal to or greater than the other side.

If it was smaller, the angle opposite

would be the smaller of the two (36), which is contrary to the hypothesis.

If it was equal to the other side, the opposite angles would be equal (35), which is likewise contrary to the hypothesis. These two suppositions being false, the only remaining one, which agrees with the enunciation, must be true.

38. *In any triangle, any side whatever is smaller than the sum of the two other sides.*

In the triangle ABC (Fig. 4), prolong BA so that AD=AC; the triangle CAD will be isosceles and the angles D and

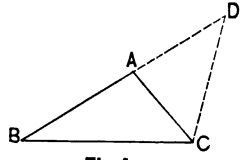


Fig.4

ACD will be equal; it follows that BCD> D, and consequently, in the triangle BCD the side BD will be greater than BC (37). But BD=BA+AC, hence the sum of the

two sides BA, AC of the triangle ABC is greater than the third. Q.E.D.

39. COROLLARY.—It results from this, *that a straight line is shorter than any broken line having the same extremities.*

In Fig. 5, we wish to prove, that the straight line A B is shorter than the broken line AEDCB. Draw the diagonals AC, AD, then by (38)

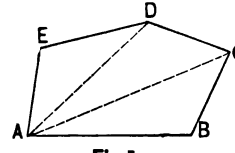


Fig.5

$$AB < BC + AC$$
$$AC < CD + DA$$
$$AD < DE + EA$$

Adding and cancelling like terms we find that,

$$AB < BC + CD + DE + EA$$

which was to be proved.

We cannot take up the subject of limits, in this short discussion; but regard-

ing any curved line as the limit of any inscribed broken line, as the length of each side becomes smaller and the number of sides increases indefinitely, we easily prove from the result above that a straight line is shorter than any curved line having the same extremities. So that we can state, generally, as a deduction from known principles, that *a straight line is the shortest between two points.*

40. It is evident now that there is no simplification, even in Legendre's method, of assuming the principle just proved; for the propositions necessary to prove the principle are theorems that should be proved in any case, hence we repeat that the unnecessary assumption should be rejected.

In fact it is very desirable to deduce all the theorems in geometry from the fewest possible assumptions, and those, too, so evident that every one recognizes their truth without hesitation; and this was plainly the aim of the older writers, who were obviously impelled to it by the many objections of the sophists; who thus

incidentally assisted in making geometry as perfect a science as it is.

It would lead us too far to allude to some other minor objections to existing discussions of certain geometrical principles; we therefore resume the consideration of the application of the general methods previously exposed.

41. The proof for the theorems just given follows the synthetical method, which gives the results so quickly that nothing better can be desired.

We shall give some applications now of the analytical method, which in many cases it is more desirable to follow than the other.

THEOREM. *Similar triangles are to each other as the squares on their homologous sides.*

We are to prove that the areas, or half the products of the bases by the altitudes, are in this ratio.

Call a and h, respectively, the base and altitude of one triangle; a' and h', the base and altitude of the other. Then we are to prove the following proportion.

$$ah : a'h' :: a^2 : a'^2,$$

which entails the following, on cancelling the common factors,

$$h : h' :: a : a';$$

and, *reciprocally*, since the first proportion can be deduced from the last.

Now the last proportion is true, since in two similar triangles the altitudes are as the sides, therefore the theorem announced is true (Article 14), since it can be synthetically demonstrated from the last mentioned known result.

Contrast the usual demonstration by deduction, as given in the books.

42. The following little exercise is taken, almost at random, from a text-book.

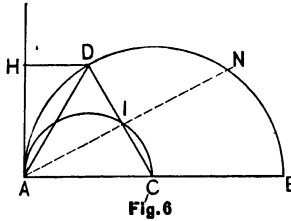


Fig. 6

Having described the semicircle AIC, on half the diameter of semicircle ABD, as shown in Fig. 6, and drawn the line CD from the center of ADB to any point D on its circumference, intersecting the smaller circumference at I, to prove that DI=DH where DH is the perpendicular from D to the common tangent at A.

Now, by analysis, we start with the relation DI=DH, we have to prove, and refer it, if possible, to another easier to prove. Thus we see at once, drawing the line AIN through I to intersection with ADNB, that AID is a right angle, being equal to AIC, an angle inscribed in the semicircle AIC; so that the two right triangles AHD and AID, having a common hypothenuse, would be equal (whence DH would equal DI) if we can only prove that the acute angles HAD and DAI are equal. We thus refer the proof of the original proposition to proving the equality of two triangles, and this in turn to proving the equality of two angles, and it is plain that the propositions are reciprocal.

Now the angles HAD and DAI are equal, the first being measured by one-half the arc AD, the second by one-half the arc DN; which arcs are equal, because CD being perpendicular to the chord AN bisects the whole arc ADN at D. Therefore the proposition is proved.

This going from the thing to be proved to known relations is much more natural, in the march of discovery, than the synthetical method in this case. By the latter method of deduction we should first notice all the relations that had been previously proved about the lines in the figure, when by combining certain of them we could deduce the theorem.

The contrast in the two methods is much more apparent, though in more complicated exercises, where a number of successive steps have to be made, but we cannot enter into them.

43. There are certain other *methods* sometimes used in the resolution of problems.

It is sometimes easier to construct a system *like* the one we wish, and after-

wards reduce it to the proper dimensions; or it may be desirable to construct a part of the new system by itself to an assumed scale and complete the construction by putting the given lines in to the same scale and afterwards reducing.

The modern graphical statics and its applications afford many examples.

44. Another expedient is the *method of geometric loci*, or finding the position of a point by tracing a line, which is the locus of the point when one condition of the problem is omitted, and noting its intersection with another line on which the point should be found likewise. The following problem is an example, though it is not the best solution of this particular problem.

Given the base = BC (Fig. 7), the length of the altitude = MH, and length = AN, of the line bisecting the angle opposite the base, from the vertex to the base, to construct the triangle.

If we draw a right-angled triangle with the altitude for one side, and the length AN for the hypothenuse, we get at once

the inclination I of this bisector to the base. The problem is to find the vertex A. It is, of course, on the line HK parallel to the base, and at the distance HM from it. Suppose the problem solved, and that ABC is the required triangle, about which circumscribe a circle,

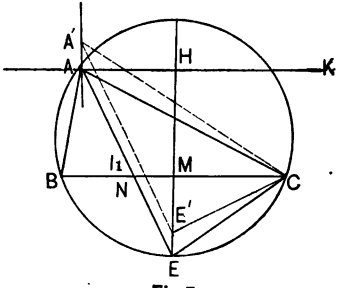


Fig.7

then extend AN to E, its intersection with the circumference, and draw EH perpendicular to BC, intersecting it at M. The last named point is the middle of BC, since arc BE=arc EC, being double the measure of the equal angles BAE and EAC respectively. Also, angle I=angle

ACE, since double their measures are arcs (AB+EC) and (AB+BE) respectively, which arcs are equal; therefore it looks probable that the last relation may give a locus for point A.

Thus, from some point E′ on the indefinite line HME′, passing through the middle of BC and perpendicular to it, draw E′A′, making the required angle I with the base, and lay off at C the angle E′CA′=I; the intersection of the dotted lines A′, will give a vertex for a triangle BA′C (not drawn entirely) that satisfies every condition but that of altitude of the required triangle.

For conceive a circle to pass through B,E′ and C; it will likewise pass through A′, otherwise it will cut the lines E′A′ and CA′ in some points F and G.

Then, since angle E′CA′ was laid off equal to I, their respective measures are equal; whence

$$\textit{arcs}\ \mathrm{GB}+\mathrm{BE'}=\mathrm{FB}+\mathrm{E'C},$$

$$\textit{arc}\ \mathrm{GB}=\mathrm{arc}\ \mathrm{BF}.$$

That is, the points F and G must coin-

cide, which is only possible when they are identical with A′, therefore the circumference that passes through BE′ and C will likewise pass through A′, hence we see that A′E′ is a bisector of the angle BA′C, so that triangle BA′C satisfies every condition but height.

It follows that taking other points as E′ on HE, and constructing the corresponding angle at C equal to I, and finding the intersection of the lines such as E′A′ and CA′ that we have a locus A′. . . for the vertex with one condition omitted. The intersection of this locus with HK produced gives A the vertex required.

45. *Application of Algebra to Geometry. Equations corresponding to Geometric Loci.*

We shall now, rather abruptly, enter the domain of co-ordinate geometry whose origin is due to Descartes, at least when the science is considered in all its generality.

Geometric loci are here represented by an equation between the co-ordinates of

each of its points, these co-ordinates being supposed to be laid off in opposite directions when they have unlike signs. Thus if $+x$ represents a distance laid off to the right, from a given origin, along a certain line; $-x$ will represent the same distance, laid off to the left, from the same origin, along the given line, a conception that forms the base of the modern algebra of algebraic numbers. Unfortunately Duhamel has made no use of this algebra, and thus is constrained to remark, after verifying the generality of an equation for both + and — values of the variables, that from an algebraic point of view, there is no sense in these operations upon minus quantities; but that applying the usual rules of multiplication of polynomials, etc., to them, they can be treated as real quantities, so far as giving the greatest generalization to the results is concerned.

This limited view of the subject we do not concur in, and refer to modern treatises on algebra, or to a previous article on algebra, by the writer, where

the theory pertaining to the algebraic series of numbers is logically developed from comprehensive definitions given to + and − in connection with elementary principles.

However, this very generality, obtained by conceiving the variables to have both plus and minus values, in the equations of geometric loci, should cause the greatest care to see that no strange solutions are introduced by too great generality, or that none are lost by introducing too many restrictions. These discussions are sometimes difficult, but if we neglect them we are not completely assured of the result.

It is with such interesting discussions that we purpose to continue the subject of the application of general methods previously exposed.

46. Let us propose first to find *the equation of the locus of points such that the ratio m of their distances to a point and a given straight line may be constant*, a condition expressed, for the following figure, by the equation,

$$AM = m.\ MP. \quad . \quad . \quad (1).$$

A representing the given point, and BD the given straight line.

Assume A as the origin, and let the axis of x be perpendicular to BD, the axis of y parallel to it, and call the distance $AB = a$.

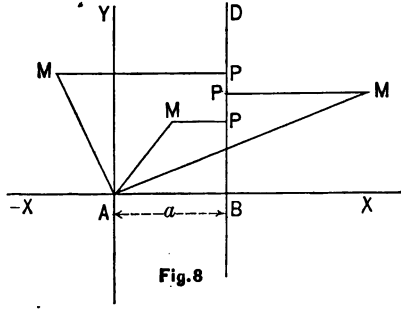


Fig. 8

Now, whether x and y are + or −, that is, wherever the point M is, we have

$$AM = \sqrt{x^2 + y^2}.$$

As to MP, when M lies between AY and BD, $MP = a - x$; when M is to the

right of BD, $MP=x-a$; finally, when M is to the left of AY, since x is essentially negative, we write, $MP = a - x$, though $(a-x)$ here is really an arithmetical sum.

The condition, eq. (1), is then replaced by the two following, the first referring to points to the left of BD, the second to points to the right of BE:

$$\sqrt{x^2+y^2}=m(a-x)$$

$$\sqrt{x^2+y^2}=m(x-a),$$

whether the point M is above or below AX.

Now if we square these equations, they can both be expressed by a single one.

$$x^2+y^2=m^2(a-x)^2 \quad . \quad . \quad (2),$$

which is thus the equation for the geometric locus required, for it includes every position of the point M, giving the corresponding value of y for any assumed value of x. There are thus no solutions lost; neither are there strange solutions; for in squaring the two equa-

tions above the result is the same if both radicals are negative as well as positive, as assumed, and the effect of changing the signs of the radicals above from + to − would simply transform the right member of one equation into the right member of the other; so that the final result (eq. 2) will be the same as before; so that equation (2) of the locus contains no strange solutions.

47. *Equation of the locus of points such that the sum of their distances to two fixed points may be constant.*

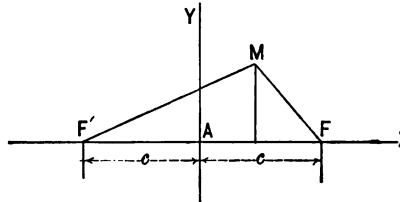


Fig. 9

Let F and F′ (Fig. 9) be the two points, $2c$ their distance apart, and $2a$ the constant sum.

Let us take for the axis of x the

straight line FF′; for the axis of y the perpendicular to FF′, erected at its middle A, and call x, y the co-ordinates of any point M whatsoever of the locus.

The condition to which it is subjected will be expressed by the equation,

$$FM+F'M=2a,$$

or,

$$\sqrt{y^2+(x-c)^2}+\sqrt{y^2+(x+c)^2}=2a \quad (1)$$

the co-ordinates having signs corresponding to the position of the point M. The formula is evidently true for any of the positions of M.

On changing the second radical to the right member, squaring and reducing, we find,

$$a^2+cx=a\sqrt{y^2+(x+c)^2}$$

Squaring again and reducing, we have

$$a^2y^2+(a^2-c^2)\,x^2=a^2\,(a^2-c^2) \quad (2)$$

Now we should get the same equation, by this repeated squaring, no matter what the signs of the radicals in (1); so that although (2) comprehends the locus

required it may contain some loci "strange" to the question.

To examine into the other solutions, we observe first that we must exclude the case of minus signs to both radicals at once; but one radical may be + and the other —, with the same result, eq. (2) above. *This corresponds to the difference of the two distances FM and F'M'*, no matter which is the greater.

We should naturally infer that (2) represents two loci at the same time, the one corresponding to the condition $FM + F'M = 2a$, the other to $FM - F'M = 2a$; but it is easily seen that this cannot be, for since $FM + F'M > FF'$ or $2a > 2c$ or $a > c$, for the first locus, in the second we must have $2a < 2c$, since in any triangle the difference of two sides is smaller than the third. Therefore if, $a > c$ we have, placing $a^2 - c^2 = b^2$

$$a^2y^2 + b^2x^2 = a^2b^2$$

for the equation of the first locus, called an *ellipse*, and if $a < c$, placing $c^2 - a^2 = b$,

$$a^2y^2 - b^2x^2 = -a^2b^2$$

for the equation of the locus of points such that the difference of their distances from F and F′ may be $2a$. a curve known as the *hyperbola.* We thus have incidentally found the equation of another locus than the one corresponding to the original enunciation. We see plainly that it is introduced by the elimination of radicals, so that eq. (2) the so-called "consequence" of (1) (see articles 19 and 20) is really more comprehensive and includes more than the consequence of the original enunciation.

48. *Equation of the locus of points such that the sum of their distances to a point and a fixed straight line may be constant.*

Let F and UV (Fig. 10) be the point and straight line respectively; AX, perpendicular to UV, the axis of x, and AY perpendicular to AX, the axis of y, drawn through a point A sufficiently far, so that all the points M that we shall consider may be on the same side of this axis.

Let x and y be the co-ordinates of any point M of the locus, situated between

UV and AY; m the given sum, FB$=a$ AF$=b$, the condition of the question is expressed by the equation FM$+$MP$=m$, or,

$$\sqrt{y^2+(x-b)^2}+(a+b-x)=m \qquad (1)$$

and we see that we evidently have $m>a$ for this case. But if we transpose $(a+b-x)$ to the right member and square both

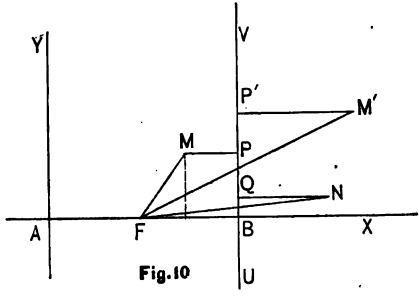


Fig.10

sides we see that we shall obtain the same result, if eq. (1) was written,

$$-\sqrt{y^2+(x-b)^2}+(a+b-x)=m,$$

which corresponds to the geometric locus, expressed by the condition, MP$-$FM

$=m$; for which, however, we evidently always have $m<a$. Rationalizing, either equation reduces to the following,

$$y^2-2(m-a)x=(m-a)^2-2b(m-a) \quad (2)$$

which is the equation of a *parabola*, which thus evidently corresponds to two loci; the one when $m>a$ answering to the original problem, the other when $m<a$, corresponding to a new geometric problem. But so far, we have only considered points to the left of UV. For points to the right $x>a+b$ and the first member of (1) becomes the difference $FM'-M'P'$. Calling this difference m, we have for points to right of UV, $m>a$, as is evident from the figure. Equation (2) which embraces this locus, also corresponds to the case when the radical is minus, *i. e.*, $FM'-M'P'=-m$; but as this has no sense, we exclude it.

Equation (2) thus corresponds to two geometric loci for points to left of UV, according as m is greater or less than a, and one locus for points to right of UV, which requires that m be greater than a.

As this equation then, for points to right of UV, does not correspond to the original enunciation, let us find the equation that does. Calling x, y the co-ordinates, of a point N such that FN + NQ = m there results,

$$\sqrt{y^2 + (x-b)^2} + x - a - b = m \qquad (3)$$

which becomes,

$$y^2 + 2(m+a)x = (m+a)^2 + 2b(m+a) \quad (4)$$

It is easy to see that eq. (3) which satisfies the geometrical condition where $x > AB$, does not satisfy it, if we take $x <$ AB, for then the first member becomes the difference FM − MP, which corresponds to the second case mentioned as belonging to eq. (2).

The problem proposed cannot then be completely resolved, except by two equations, and each of the two represent loci, of which a part belongs, whilst the other does not belong to the question, and yet it is impossible to separate them.

This interesting example should teach caution in assuming that an equation

deduced to represent the position of certain points is necessarily general or corresponds to all points of the locus.

49. In the preceding cases the strange solutions have been introduced by the elimination of radicals. They may likewise be found when an equation contains a term of the form $\frac{F(x,y)}{\varphi(x,y)}$, and we multiply both members by $\varphi(x,y)$. The new equation will admit necessarily of values that annul $\varphi(x,y)$ and $F(x,y)$ at the same time. If these solutions pertain to the question, there are no strange solutions introduced by the multiplication; otherwise there will be, and it will be impossible to suppress them.

A case in point is the equation of the conchoid.

$$\frac{y}{\sqrt{y^2+x^2}}=\frac{y-a}{m} \quad . \quad . \quad (1)$$

Making $x=o$, we find $y=a\pm m$; but if we clear of fractions, the values $x=0$, $y=o$ satisfy the equation, but not the geometric condition (as is well known),

and is thus strange to the question. The left member of eq. (1) takes the form $\frac{0}{0}$ for these values.

50. *Locus of all the points from which we can view, under the same angle, a straight line of given length and position.*

Let BB′ (Fig. 11) be the given straight line, $2a$ its length, and t the tangent of

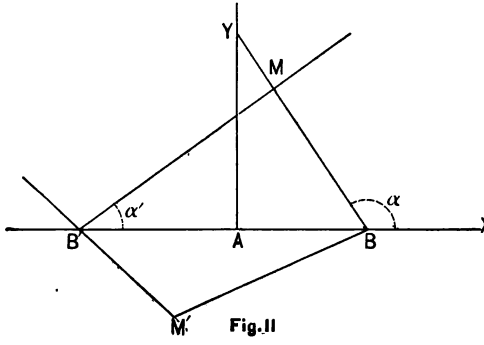


Fig. 11

the angle under which it must be viewed from any point M of the locus.

Take BB′ for the axis of x, and the

perpendicular erected at its middle for the axis of y.

The equations of the straight lines drawn from B and B′ to M, calling m and m' the tangents of a and a' and a and a, the intercepts on the axis of x, are,

$$y=m(x-a),\ y=m'(x+a).$$

If the point M is above the axis of x, the angle under which BB will be seen is $(a-a')$, and we have

$$t=\frac{m-m'}{1+mm'} \quad . \quad . \quad . \quad (1)$$

But if the point is at M′, below AX, the angle under which BB′ will be seen will be $(a'-a)$, whose tangent is,

$$\frac{m'-m}{1+mm'}=-t.$$

The points of the plane *satisfying* (1) will be such then, that from those above AX we shall see BB′ under the angle whose tangent is t, whereas from those below AX, BB′ will be seen under the angle whose tangent is $-t$, that is under the supplementary angle. We make this

explanation now, as we necessarily include this result in making use of the single equation (1), and cannot get rid of it. Finding the values of m and m' from the equations of the straight lines and substituting in equation (1) and reducing, we find,

$$y^2+x^2-a^2=\frac{2ay}{t};$$

which can be put under the following form,

$$\left(y-\frac{a}{t}\right)^2+x^2=a^2\left(1+\frac{1}{t^2}\right) \quad . \quad (2)$$

which we recognize at once as the equation of a circle, whose centre has $x=o$, $y=\frac{a}{t}$, and whose radius is $=\frac{a\sqrt{(1+t^2)}}{t}$.

Its upper and lower parts are segments capable of supplementary angles, as we have foreseen.

The locus of points below AX, from which BB′ would be viewed under the given angle whose tangent is t, would correspond to eq. (1) if t were replaced in it by $-t$, and both the resulting equa-

tion and (2) above would correspond to circles, of which only a part belongs to the question proposed.

We have here an example of lost and strange solutions at the same time, for eq. (1) does not include the locus of *all* the points above and below AX, from which BB′ can be viewed under the same angle, so that by Article 17 there will be *lost* solutions; but we recognized at the start, also, that eq. 1 satisfies two different geometrical conditions for points above and below AX, so that all the solutions of eq. 1 are not solutions of the original problem or the conditions are not reciprocal, whence by the reasoning of Article 19 the *strange* solutions are accounted for.

We see here again how erroneous it would have been to have assumed that an equation deduced by considering one point only, M above the axis of x, was general and applicable to all points of the required locus.

An attentive consideration of the preceding examples, in connection with the

exposition of general methods, previously given, shows that great care must be exercised, in endeavoring to find a single equation to represent a given geometric locus, in order to avoid lost or strange solutions, which will surely be found if the conditions of the two problems are not reciprocal.

51. The converse problem, that of finding the geometric locus corresponding to a given equation, is often simpler, for here there are no lost or strange solutions, but only those that can regularly be deduced from the equation; but even in this case we must consider both plus and minus values of the quantities that admit of them entering the equation; for when an equation is given we tacitly assume it to be general, unless it is expressly stated that it pertains only to certain values or quadrants or other limits. But this very generality, without which analytical geometry could not exist as a complete science, demands a corresponding circumspection in all the discussions pertaining to it.

This observation naturally suggests certain remarks relative to formulæ expressed in terms of polar co-ordinates, with which we shall conclude this discussion.

52. If we consider a point in a plane referred to rectangular co-ordinates x, y, the formulæ for passing to polar co-ordinates are

$$x = r\cos\varphi,\ y = r\sin\varphi \quad . \quad . \quad . \quad (1).$$

where r = radius vector from the origin of rectangular co-ordinates and φ, the angle made by it with the axis of x. Now, since $cos\ \varphi$ is positive or negative with x, and $sin\ \varphi$ positive or negative at the same time as y, it is evident that the above formulæ are general, and hold for any values of the angle φ, provided we take the signs of the functions established in trigonometry, and regard x and y as positive in one sense and negative in the other, r, being regarded always as positive and in absolute value. But a little inspection of the formulas will show that if we put $(\varphi \pm \pi)$ for φ, and $-r$

for r, that the formulas will pertain to the same point, if we agree that r must be laid off in an opposite direction from the origin to the radius vector, since r and the trigonometric functions both change signs at the same time, so that the products giving the values of x and y will be the same as before. (Puckle, in his *conic sections*, page 8, has assumed this principle, without deriving it from any formulas, and given illustrations of its application.)

Now let the equation of a locus expressed in rectangular co-ordinates be

$$F(x, y)=0;$$

its polar equation will be

$$F(r\cos\varphi, y\sin\varphi)=0 \quad . \quad . \quad (2)$$

and we have seen that, taking positive values of r corresponding to any angle φ, eq. (2) will give all the points of the locus represented by the eq.

$$F(x, y)=0;$$

which we should likewise find by substituting $(\varphi \pm \pi)$ for φ at the same time that we put $-r$ for r, since the same

point of the locus is given by the co-ordinates $(\varphi \pm \pi)$, and $-r$ as by φ and $+r$, with the distinct understanding that all angles are to be laid off, going around in the same direction from the axis of x, and that $-r$ is to be laid off on the radius vector produced and of the same absolute value as $+r$. Thus we can either regard r as always positive, and vary φ from 0 to 2π to get all the points, or admitting negative values of r it will suffice to vary φ from 0 to π.

53. It is important to observe that if the original equation in rectangular co-ordinates $F(x, y)=0$, was subjected to any restriction, that Eq. (2) will likewise be subjected to the same restriction.

To illustrate, suppose $F(x,y)$ contains a radical of the form $\sqrt{mx^2+ny^2}$, that must always be regarded as positive. Transforming this into polar co-ordinates by means of eqs. (1), the radical becomes,

$$+\sqrt{m r^2 \cos^2 \varphi + n r^2 \sin^2 \varphi},$$

and will remain the same when we put $\varphi + \pi$ and $-r$ in place of φ and r.

Now, if we had passed the factor r^2 outside of the radical and written it

$$r\sqrt{m\cos^2\varphi + x\sin^2\varphi},$$

we must always take r positively, for changing φ and r to $\varphi+\pi$ and $-r$, would give a minus radical. The double construction then only applies in this case when the radical can be taken with the double sign.

54. Again, when the transformed equation (2) can be decomposed into factors, the double construction will not apply necessarily to each factor separately, for it has not been so demonstrated.

Thus the polar equation of the ellipse, with the focus for a pole, is

$$(a^2-c^2\cos^2\varphi)\,r^2+2b^2cr\cos\varphi-b^4=0,$$

and we see that if r and φ satisfy the equation, $-r$ and $\varphi+\pi$ will likewise satisfy it; but if we decompose the first member into two factors, and put them equal to zero, we find the two equations,

$$r=\frac{b^2}{a+c\cos\varphi},\quad r=\frac{b^2}{-a+c\cos\varphi},$$

neither of which is satisfied when we replace r and φ by $-r$ and $\varphi+\pi$, although one is converted into the other.

In constructing the locus from either one of these equations, we notice that the first always gives positive values of r, whereas the second will always give negative values of r, since $a>c$; therefore if we use the last equation we must be careful to lay off the values of r in an opposite sense to those for $+r$. The values of the last will agree with those pertaining to the first equation, when φ in the first corresponds to $\varphi+\pi$ in the second, as we have seen.

A similar investigation can be made for the hyperbola, with corresponding results.

55. Where the angle φ, and not its trigonometric function, enters an equation, the previous reasoning does not apply. Let us take a simple example, the *spiral of Archimedes*, and ascertain if any generalization can be effected by the use of minus radii vectors, and whether a double construction is possible.

Suppose a point moving uniformly along a straight line which turns uniformly around one of its points in a given plane.

Let us take for a pole the fixed point A, and for an axis AU; call AB$=a$, AM$=r$, and MAB$=\varphi$; then if b designates the amount the point moves along the straight line, whilst it turns an angle equal to unity, taking for the measure of the angles at the center the ratio of the arcs

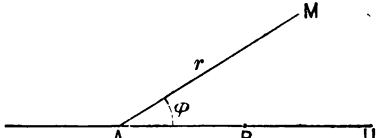


Fig. 12

intercepted to the radius, we have,

$$r=a+b\varphi \quad . \quad . \quad (1)$$

for the equation of the locus, if $a=$ AB gives the value of r on AU when $\varphi=0$.

This equation holds for all values of φ from 0 to ∞. But can it represent the positions of the moving point anterior to

its arrival at B? We can answer this question in the affirmative, if we agree that angles counted in the opposite direction to φ shall be minus, for then the equation above takes the form $r = a - b\varphi'$, if φ' is the absolute value of the angle reckoned to the right.

If this turning in the negative direction is continued, so that $b\varphi'$ becomes greater than a, the same equation suffices to represent the position of points in the indefinite past, before A was reached, provided we agree to lay off the minus values of r in an opposite direction to the plus values. The equation now takes the form $-r = b\varphi' - a$.

With the conventions established we see that negative values of r and φ serve to generalize formulas in equations where the angle, and not its function, enters; so that one formula, (1) in this instance, suffices to represent all the points of the locus.

But the double construction, previously mentioned, for the same point, does not hold in this case. Thus equation (1)

is not satisfied when we change the coordinates φ and r to $\varphi + \pi$ and $-r$.

It is seen from the preceding investigation that a negative radius vector is not always admissible, but that when it is, there can be no possible objection to its use, especially if an investigation is thereby simplified in any way.

In fact the use of both plus and minus quantities, in the sense of opposition throughout the whole range of mathematics, allows a condensation in the expression of results, without which the science would be in a very imperfect state; but this very contrivance imposes a corresponding attention to every phase of a question, in order to establish the generality, when it exists, or the proper limitations when it does not exist.

It is hoped that the preceding exposition and illustrations of the methods to be used will prove of assistance in such researches.

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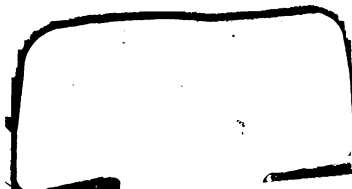
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